Geometric Function Theory
Explorations in Complex Analysis

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Complex analysis is a rich and textured subject. It is quite old, and its history is broad and deep. Yet the basic graduate course in complex variables has become rather cut and dried. The choice of topics, the order of the topics, and the overall flavor of the presentation are strongly influenced by the need to prepare students for the qualifying exams. The qual-level course is designed to serve a limited purpose, and it does that but little more.

Basic complex analysis is startling for its elegance and clarity. One progresses very rapidly from the basics of the Cauchy theory to profound results such as the fundamental theorem of algebra and the Riemann mapping theorem. Many a student is left, at the end of the course, yearning for more—to be advanced to a level where he or she could consider research questions, indeed the possibility of writing a thesis in the subject.

Yet there are few places to turn in such a quest. The book [BEG] of Berenstein and Gay and the book [CON] of Conway give particular takes on some of the more advanced material in the subject. Some of the older books, such as [FUKS], [GOL], [HIL], [MAR] treat topics not usually found in the basic texts. But it seems that there is a need for a book that will open the student’s eyes to what this subject has to offer, and give him or her a taste of some of the areas of current research. This is meant to be such a book.

Our prejudice in the subject is geometric, but this does not prevent us from exploring byways that come from analysis, algebra, and other parts of mathematics. Thus, on the one hand, we treat invariant geometry, the Bergman metric, the automorphism groups of domains, and the boundary regularity of conformal mappings. On the other hand, we also explore the Hilbert transform, the Laplacian, the corona problem, harmonic measure, the inhomogenous Cauchy–Riemann equations, and sheaf theory.

The aim of the book is to expose the student to mathematics as it is practiced: as a synthesis of many different areas, exhibiting particular flavors and features that arise from that synthesis. The student who reads this book should be inspired to go further in the subject, to begin to explore the primary literature, and (one hopes) to think about his or her own research problems.
One of the rewards and pleasures that the student will find in reading this book is the rich interactions that are displayed among the various topics. For example, harmonic measure is used to prove a sharp version of the Lindelöf principle. It is also used to establish the three lines (viz. three circles) theorem of Hadamard. This in turn is used to prove (in another chapter) the Riesz–Thorin theorem. In another venue, the Riesz–Thorin theorem is used to prove the $L^p$-boundedness of the Hilbert transform. The Laplacian is reviewed and used as a device to introduce the Green’s function. The Green’s function is used, of course, to derive the Poisson kernel. But it is also used to develop the Bergman kernel. The Bergman kernel is used to study boundary regularity of conformal mappings. It is also used in the study of automorphism groups, and to prove various uniqueness theorems for conformal mappings. The Bergman metric interacts with and arises alongside consideration of other conformal metrics, and leads to Ahlfors’s version of the Schwarz lemma.

The Poisson kernel is used to study the boundary behavior of harmonic and holomorphic functions. The Dirichlet problem for the Laplacian is used to give a nonstandard proof of the Riemann mapping theorem. The Green’s function is used to prove a canonical representation of multiply connected regions on slit-domains. The Ahlfors map gives a new view of uniformization as introduced by these topics.

The Green’s function and Stokes’s theorem lead to a solution of the inhomogeneous Cauchy–Riemann equations, which are in turn used to make new constructions in function theory. The Green’s function and the Hilbert transform, together with our solution of the inhomogeneous Cauchy–Riemann equations and the F. and M. Riesz theorem, are used to derive a proof of the corona theorem. Duality properties of Hardy spaces (covered in an earlier chapter) are exploited along the way.

Our study of the Ahlfors map uses Banach algebra properties of $H^\infty$ that we developed in our study of the corona theorem. It also gives a reprise of the Green’s function and harmonic measure. The Green’s function and Green’s theorem are used extensively in the proof of the corona theorem and also in our development of the uniformization theorem. Nontrivial ideas from functional analysis arise in our study of the Riesz–Thorin theorem, the Hilbert transform, the summation of Fourier series, the corona theorem, and the Ahlfors map.

We provide a discussion of the statement, concept, and proof of Köbe’s uniformization theorem, together with various planar variants. Thus we examine uniformization from many different points of view. When we treat automorphism groups, uniformization provides a powerful tool.

Algebra is encountered in various guises throughout the book. Certainly it plays a role in the group-theoretic aspects of automorphisms. It occurs again in our treatment of Banach algebra techniques. And it plays a decisive role in the study of sheaves. Sheaf theory gives the student a new way to view the Weierstrass and Mittag–Leffler theorems, as well as questions of analytic continuation. Thus this text shows the students many different aspects of complex analysis, and how they interact with each other.
With this book, the student and the advanced worker too are introduced to a rich tapestry of function theory as it interacts with other parts of mathematics. There is hardly any other analysis text that offers such a variety and synthesis of mathematical topics.

Part of my own training is to think of the subject of complex analysis as a foil, or perhaps as a gadfly. Many an interesting problem of geometric analysis is very naturally formulated in the language of complex function theory; but then it is best solved by stripping away the complex analysis and applying tools of geometry, or partial differential equations, or harmonic analysis. That is the point of view that we shall, at least in part, take in the present book. Complex analysis will be our touchstone, but it will be the entrée to many another byway of mathematics—from the Cauchy–Riemann equations to interpolation of linear operators to the study of invariant metrics. It is our view that this is a productive and rewarding way to practice mathematics, and we would like to teach it to a new generation.

It is a pleasure to thank my editor, Ann Kostant, for encouraging me to write this book, and for making the process as smooth and carefree as possible. She enlisted strong and insightful reviewers to help me craft this book into a more precise and useful tool. I thank Elizabeth Loew for a marvelous job of editing and TEX typesetting. I look forward to comments and criticisms from the readership, and hope to make future editions more accurate and therefore more useful.

St. Louis, Missouri and Berkeley, California

Steven G. Krantz
Part I

Classical Function Theory
Overview

The subject of the function theory of several complex variables is strictly an artifact of twentieth-century studies. Its hallmark has been the introduction of powerful and abstract methods from algebra, algebraic topology, partial differential equations, harmonic analysis, Lie groups, differential geometry, and many other parts of modern mathematics. Although we shall not treat several complex variables as such in the present text, we shall present some aspects of classical one-variable complex function theory that may serve as an entrée to the multidimensional techniques. These will include automorphism groups and cohomological topics.

The treatment in these two chapters will share with the reader some non-standard parts of our subject. Again, they illustrate the symbiosis with other parts of mathematics that complex analysis has enjoyed for much of its history. Complex variable theory is an elegant and dynamic subject that is an exciting source of problems. To actually solve those problems, it is often most efficacious to strip away the complex variable theory and bring in tools from another part of mathematics.
Invariant Geometry

1

Genes11 and Development

The idea of using invariant geometry to study complex function theory has its foundation in the ideas of Poincaré. Certainly he is credited with the creation of a conformally invariant metric on the unit disk $D$. The uniformization theorem (covered later in this book) may be used to transfer the metric to other planar domains. Later on, Stefan Bergman found a way to define invariant metrics on virtually any domain in any complex manifold. We shall explore his ideas further on in the book.

The geometric approach provides a new way to view the subject of complex variables. It is the source of tantalizing new questions. But it also provides a vast array of powerful new weapons to use on traditional problems. Any number of problems about mappings and conformality—just as an instance—are rendered transparent by way of geometric language.

An appreciation of the concepts in this chapter requires understanding of the idea of a Riemannian metric. Our presentation, however, is self-contained. The reader may learn what such a metric is by doing, that is, by studying this chapter. Even the student who is new to geometric language will find that the material in the ensuing pages will introduce him to a new way to approach function theory.

1.1 Conformality and Invariance

Conformal mappings are characterized by the fact that they infinitesimally (i) preserve angles, and (ii) preserve length (up to a scalar factor). It is worthwhile to picture the matter in the following manner: Let $f$ be holomorphic on the open set $U \subseteq \mathbb{C}$. Fix a point $P \in U$. Write $f = u + iv$ as usual. Thus we may write the mapping $f$ as $(x, y) \mapsto (u, v)$. Then the (real) Jacobian matrix of the mapping is